MATCHACURVE-2 FOR ALGEBRAIC TRANSFORMS
TO DESCRIBE
CURVES OF THE CLASS $X^n$

Chester E. Jensen and Jack W. Homeyer
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THE AUTHORS

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ABSTRACT

Graphed curves of the class, $X^n$, can be quickly scaled to and compared with the wide array of standard curves presented. If a standard can be found that suitably matches the scaled curve, the algebraic transform specified for the standard can be applied to the X values of a relevant data set and these can be scaled to their corresponding Y values by least squares to arrive at the fitted model.
INTRODUCTION

The analyst often initiates a two-dimensional regression analysis by hand-fitting the expected curve for an XY relation through a set of plotted data points called the "graphed curve" in this paper.

Assume he wants next to find a mathematical form of the independent variable, X, which when scaled to the graph emulates the graphed curve with acceptable accuracy. If the curve is linear, X itself is, of course, the appropriate form. But if the curve is nonlinear, a matching nonlinear form of X should be the object of his search. Once having found a suitable form, the analyst can rescale it to the actual data points of any relevant data set\(^1\) by least squares, and the development of the estimating equation is complete.

In this paper we have attempted to reduce the effort required to find acceptable transforms of the \(X^n\) class. The analyst simply compares a scaled version of his graphed curve to graphed Standards (page 9) and selects the two adjacent ones most nearly like his own in shape, a process hereinafter referred to as "Matchacurve."

Interpolation between the transforms given for these Standards results in identification of the best alternative offered by the system. It is this transform of \(X\) that is finally fitted to a relevant data set by least squares.

Three unique sets of standard curves are presented. Each set is based on a selected array of \(n\) in \(X^n\), with limits as shown:

\[
\begin{array}{|c|c|}
\hline
\text{Set} & n\text{-array limits} \\
\hline
1 & 1.00 \leq n \leq 20.00 \\
2 & 0.10 \leq n \leq 1.00 \\
3 & -2.00 \leq n \leq -0.01 \\
\hline
\end{array}
\]

Each set of Standards is transformed as required (see Appendix C) to appear in each of the four basic positions in which the analyst's curve can occur in the upper right quadrant so that there are 12 sets of Standards in all--3 sets in each of 4 positions.

\(^1\)Either the data set from which the graphed curve is derived or some other set to which this curve is judged applicable.
The scaling procedure for sets 1 and 2 is shown in the example that follows. Set 3, because it involves reciprocals and because the latter are indeterminant at $X = 0$, requires that $Y_p$ be determined within a slightly abbreviated $X$-range as explained in the last paragraph under the section labeled "An Example."

For the sake of efficiency, then, it is suggested that the analyst's curve be scaled first for comparison to sets 1 and 2; the curve will be scaled for comparison to set 3 only when no suitably similar form can be found in sets 1 and 2.

Where the analyst's curve lies either partly or wholly in some quadrant other than "upper right" (i.e., not oriented as the curves above), adjust either the $X$- or $Y$-scale, or both, to put the curve on the same basis as the standards before applying Matchacurve --see Appendix A.

**AN EXAMPLE**

This example applies as shown to sets 1 and 2.

---

**Figure 1.**—Here, the expected curve form has been hand-fitted through plotted data points. Plus and minus departures are balanced approximately.

Let's say that an analyst wants to generate an equation for $Y$ from the data set plotted in figure 1. From previous knowledge of the relation between $X$ and $Y$, he expects the curve to be concave upward with larger values of $X$. The apparent data trend supports his expectation and he hand-fits such a curve through the data, also shown in

---

2This scaling procedure is identical to that employed for the 1970 companion paper for sigmoids, Matchacurve-1, so the same scaled curve can be compared to the sigmoid standards, if pertinent.
figure 1. This curve then, is to be emulated using the Matchacurve process detailed in
the steps below:

1. Let \( X_p \) be the value of \( X \) at or near the largest \( X \)-value in the data set. Let
\( Y_p \) be the value of \( Y \) at \( X_p \). Determine the values of \( X_p (=800) \) and \( Y_p (=66) \) as shown in
figure 1. Note that for sets 1 and 2, \( Y_p \) is measured at \( X = 0 \) or at \( X_p \) depending on
position--see table 1. Select representative point coordinates from the smoothed curve.

Five points have been chosen for this example.\(^3\) Call the array of \( X \) values \( X_1 \), and
the array of \( Y \) values, \( Y_1 \).

\[ \text{Point coordinates from the smoothed curve.} \]

\[
\begin{array}{c|ccccc}
X_1 & 0 & 200 & 400 & 600 & 800 \\
Y_1 & 0 & 1.3 & 10.6 & 31.4 & 66 \\
\end{array}
\]

2. Scale the \( X_1 \) to \( X_1/X_p \) and the \( Y_1 \) to \( Y_1/Y_p \).

\[ \text{Scaled coordinates} \]

\[
\begin{array}{ccccc}
X_1/800 & 0.00 & 0.25 & 0.50 & 0.75 & 1.00 \\
Y_1/66 & 0.00 & 0.02 & 0.16 & 0.48 & 1.00 \\
\end{array}
\]

3. Plot these points on a sheet of graph paper at the exact scale\(^4\) shown in
figure 2. Draw a smooth curve through the points.

4. From table 1, identify the "position"--position A, here--of the scaled curve
in figure 2. Then use the scaled curve as an overlay for Set 1 and Set 2 of the
Standards, position A; be certain that the \( X \) and \( Y \) axes are matched exactly. Find the
Standards that bracket the overlay curve of figure 3.\(^5\) In this case, \( n = 2.5 \) and 3.0 of
Set 1 bracket the overlay curve nicely. Use proportional departure of the overlay curve
from the left bracketing Standard to approximate an interpolated value between \( n = 2.5 \)
and 3.0. This is best done at a point where curve form accuracy is most crucial--the
point of sharpest bend was adopted in this case. From figure 3 the interpolated \( n \) is
2.7.

---

\(^3\)Use fewer points where the desired sensitivity is less, more for greater
sensitivity.

\(^4\)This is the same scale used for the Standards. Note that the odd \( X \) scale was
necessitated by publication limitations on paper size.

\(^5\)A light source behind the graphs that are being compared will assist in the
matching process.
Table 1.—Information on the Standards (Page 2)

<table>
<thead>
<tr>
<th>A</th>
<th>B/A</th>
<th>Exponent range</th>
<th>Tp predicted</th>
<th>X transformed to be fitted by least squares</th>
<th>Sign of B/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0 ≤ x ≤ 20.0</td>
<td>x_p</td>
<td>(x - x_p)^b</td>
<td>-</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>0.1 ≤ x ≤ 1.0</td>
<td>x_p</td>
<td>(x - x_p)^b</td>
<td>-</td>
<td>*</td>
</tr>
<tr>
<td>3</td>
<td>-2.0 ≤ x ≤ -0.01</td>
<td>x_p</td>
<td>(x - x_p)^b</td>
<td>-</td>
<td>*</td>
</tr>
</tbody>
</table>

For achieving the specified positions for some of the sets, the x-scale must be reversed before applying the exponents as indicated here.

CAUTION: Limits of use for Matheson's transforms are as follows:

Sets 1 and 2, 0 ≤ x < x_p
Set 3, 0.01 x_p ≤ x ≤ x_p

Figure 2.—Here, the graphed curve has been scaled to the Standards.
5. Substitute $Y$ for $Y/Y_p$ in the final, simple linear model to be fitted to a relevant data set (the original set is used here) by least squares:

$$ Y = B_0 + B_1 X^{2.7} $$

Then the estimated $B_0$, $B_1$ -- or $\hat{B}_0$, $\hat{B}_1$ -- will scale the $X_i$ transforms to the $Y_i$.

It is generally preferable to use a computer for transforming the $X_i$ and applying the least squares fitting process and, in this case, the fitted model is:

$$ \hat{Y} = 2.06 + (0.8932 \times 10^{-6}) X^{2.7}, \ 0 \leq X \leq X_p $$

The curve for this model is compared to the original graphed curve in figure 4. Differences indicate the extent to which the hand-fitted curve failed to follow the least squares path. We assume here that the least squares curve is the most desirable alternative within the $X$-range of the data.

---

Using logarithm tables for the $X^{2.7}$ transforms and a desk calculator for the fitting process, the more laborious hand calculations would be as shown in Appendix B.
Figure 4.--The fitted model has been plotted over the original graphed form.

The foregoing procedures apply in detail to set 3 with the exception that $Y_p$ is measured as shown in the graphs below:

<table>
<thead>
<tr>
<th>Position</th>
<th>Analyst's curve</th>
<th>$Y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><img src="image" alt="Graph A" /></td>
<td>$Y_p$ $Y_{X_p} - Y_{0.01X_p}$ *</td>
</tr>
<tr>
<td>B</td>
<td><img src="image" alt="Graph B" /></td>
<td>$Y_{0.01X_p}$</td>
</tr>
<tr>
<td>C</td>
<td><img src="image" alt="Graph C" /></td>
<td>$Y_{X_p} - Y_{0.01X_p}$</td>
</tr>
<tr>
<td>D</td>
<td><img src="image" alt="Graph D" /></td>
<td>$Y_{0.01X_p}$</td>
</tr>
</tbody>
</table>

* $Y_{X_p} = Y$ at $X_p$, $Y_{0.01X_p} = Y$ at $0.01X_p$
The portion of the curve being "matched" in every position is that over the X-range from 0.01X₀ through X₀. Be sure to match this X-range of the analyst's curve with the corresponding X-range of the Standards. Adopted transforms from this set apply only within the limits 0.01X₀ ≤ X ≤ X₀.

MISCELLANEOUS NOTES ON MATCHACURVE

• There will be cases where the X transform determined from Matchacurve is not sufficiently accurate to suit the analyst's purposes. In such cases, curve-form description procedures beyond the scope of this paper must be used. For example, refer to nonlinear regression in the text by Draper and Smith.7/

• The alternative of using a completely automated X-transform selection system from the Matchacurve family of curves was considered. Such a system was bypassed in favor of the graphic comparison technique, which minimizes constraints on curve selection criteria and is still relatively fast to use. It is important to note that selection criteria vary with analysts, analytical objectives, and data characteristics. Our inability even to define applicable criteria commonly used (aside from least squares) discouraged the adoption of complete automation here.

---

STANDARDS
A-1.--Standards for position A, set 1, -- (X-transform to be fitted by least squares = $(X)^n$, $0 \leq X \leq X_p$)
A-2. Standards for position A, set 2—(X-transform to be fitted by least squares = \((X_p - X)^n\), 0 \leq X \leq X_p)
A-3. -- Standards for position A, set $\beta_0$ (X-transform to be fitted by least squares = $(X_p - X)^n$, $0.01X_p \leq X \leq X_p$)
B-1.--Standards for position B, set 1-n (K-transform to be fitted by least squares = \((X_p - X)^n\), \(0 \leq X \leq X_p\))
B-2. Standards for position B, set 2, \( (X) = (X)^n \), \( 0 \leq X \leq X_p \)
B-3. --Standards for position B, set 3 -- (X-transform to be fitted by least squares $= (X)^n$, $.01X_p \leq X \leq X_p)$
C-1. -- Standards for position C, set 1 -- (X-transform to be fitted by least squares = \((X_p - X)^n\), \(0 \leq X \leq X_p\))
C-3.--Standards for position C, set 3--(X-transform to be fitted by least squares = \( (X)^n \), \( 0.01X_p \leq X \leq X_p \))
D-3. Standards for position D, set \( s \) -- (X-transform to be fitted by least squares = \( (X_p - X)^n \), \( .01X_p \leq X \leq X_p \))
APPENDIX A
ADJUSTMENT OF THE ANALYST'S CURVE TO THE UPPER RIGHT QUADRANT

If the curve lies partly or wholly to either side of X = 0, change the X values of point coordinates from the curve to X' = (X - X minimum)--for example:

![Graphs showing adjustment of X coordinates](image)

<table>
<thead>
<tr>
<th>X</th>
<th>X' = X - (-30)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-30</td>
<td>0</td>
</tr>
<tr>
<td>-20</td>
<td>10</td>
</tr>
<tr>
<td>-10</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>X' = X - (-10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>X' = X - (10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

Plotted over X', all three curves would now appear as follows:

![Graph showing adjusted coordinates](image)
If the curve lies either partly or wholly above or below \( Y = 0 \), change the \( Y \) values of point coordinates from the curve to \( Y' = (Y - Y_{\text{minimum}}) \)—for example:

\[
\begin{array}{c|c}
Y & Y' \\
-5 & 0 \\
-3 & 2 \\
-1 & 4 \\
\end{array} \quad \begin{array}{c|c}
Y & Y' \\
-3 & 0 \\
-1 & 2 \\
1 & 4 \\
\end{array} \quad \begin{array}{c|c}
Y & Y' \\
1 & 0 \\
3 & 2 \\
5 & 4 \\
\end{array}
\]

Plotting \( Y' \) instead of \( Y \), all three curves would appear as:

And, last, \( X' \) and \( Y' \) would be substituted for \( X \) and \( Y \), respectively, when looking for a suitable form in the Standards.
In fitting an adopted form to a relevant data set by least squares, always revert to the original $Y_i$ values—but retain any X-transforms in the fitted model. Assume, for instance, that the use of $X' = (X + 30)$ and $Y' = (Y + 3)$ had been necessary to orient the analyst's smoothed curve in Figure 1 to the upper right quadrant as shown. The Y-axis would then be labeled $Y'$ and the X axis, $X'$. All procedures would be identical to those shown for the example except that the final model fitted would be:

$$Y = \hat{B}_0 + \hat{B}_1 (X + 30)^{2.7}$$

--and, $\hat{B}_0$ should approximate the constant added to Y in the adjustment process, or, $\hat{B}_0 \approx 3$.

**APPENDIX B**

**LEAST SQUARES FIT OF EXAMPLE MODEL, HAND CALCULATIONS**

<table>
<thead>
<tr>
<th>Observation number</th>
<th>Original data</th>
<th>Let $X^{2.7}$</th>
<th>$X'Y$</th>
<th>$(X')^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$X$</td>
<td>$Y$</td>
<td>$X' = \frac{X}{0.163229 \times 10^6}$</td>
<td>$X'Y$</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
<td>1</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>2</td>
<td>300</td>
<td>1</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>3</td>
<td>370</td>
<td>13</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>4</td>
<td>400</td>
<td>24</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>5</td>
<td>500</td>
<td>5</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>6</td>
<td>530</td>
<td>30</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>7</td>
<td>600</td>
<td>18</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>8</td>
<td>650</td>
<td>30</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>9</td>
<td>660</td>
<td>61</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>10</td>
<td>730</td>
<td>60</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
<tr>
<td>11</td>
<td>750</td>
<td>40</td>
<td>0.163229 $\times 10^6$ etc.</td>
<td>0.163229 $\times 10^7$ etc.</td>
</tr>
</tbody>
</table>

$\Sigma$ $X = 283$  \hspace{1cm} $\Sigma X' = 0.2915176 \times 10^9$  \hspace{1cm} $\Sigma X'Y = 0.1094696 \times 10^{11}$  \hspace{1cm} $\Sigma (X')^2 = 0.1158486 \times 10^{17}$

Mean $\Sigma X = 25.727$  \hspace{1cm} $\Sigma X' = 0.2650160 \times 10^8$

Continuing the computations:

$\Sigma (X')^2 = \Sigma (X')^2 - (\Sigma X')^2/n = 0.3859179 \times 10^{16}$

$\Sigma (X'Y) = \Sigma (X'Y) - (\Sigma X' \Sigma Y)/n = 0.3447002 \times 10^{10}$

$\hat{B}_1 = \frac{\Sigma (X'Y)}{\Sigma X^2} = 0.8931957 \times 10^{-6}$

$\hat{B}_0 = \bar{Y} - \hat{B}_1 \bar{X}' = 25.72 - \hat{B}_1 (0.2650160 \times 10^8) = 2.06$

and

$\hat{Y} = 2.06 + (0.8932 \times 10^{-6}) X^{2.7}$
APPENDIX C

DOCUMENTATION OF STANDARDS

Curves of the three exponential sets are oriented in the upper right quadrant as shown below:

\[
\begin{align*}
\text{SET} & \\
1 & \quad 1.0 \leq n \leq 20.0 \\
2 & \quad 0.1 \leq n \leq 1.0 \\
3 & \quad -2.00 \leq n \leq 0.01
\end{align*}
\]

Each of these basic curve types above is presented in each of the four possible positions for the analyst's curve in the upper right quadrant.

When the inherent exponential curve position differed from A, B, C, or D, it was necessary to reorient the exponential curve by reversing X-axes and/or rotating scaled X-transforms about Yp'.
The net result was the series of transforms scaled to 1.0 in both \( X \) and \( Y \) as indicated below:

<table>
<thead>
<tr>
<th>Position</th>
<th>Set</th>
<th>Plotted form*</th>
<th>Plotting range in ( X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>( Y = X^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Y = 1-(1-X)^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>( Y = \left[(1-.01^n-1)^{-1}\right] [(1-X)^n-1] )</td>
<td><strong>0-.99</strong></td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>( Y = (1-X)^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Y = 1-X^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>( Y = [(.01^n-1)^{-1}] (X^n-1) )</td>
<td>.01-1.00</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>( Y = 1-(1-X)^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Y = X^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>( Y = 1-[(.01^n-1)^{-1}] (X^n-1) )</td>
<td>.01-1.00</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>( Y = 1-X^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Y = (1-X)^n )</td>
<td>0-1.00</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>( Y = 1-[(.01^n-1)^{-1}] [(1-X)^n-1] )</td>
<td><strong>0-.99</strong></td>
</tr>
</tbody>
</table>

*These forms were submitted to a computer-controlled plotter to produce the Standards in this paper.

**Although these Standards are actually plotted over the range 0-.99, they are applicable to the range .01-1.00 and were so labeled.